REFLECTION GROUPS ACTING ON THEIR HYPERPLANES

IVAN MARIN

Institut de Mathématiques de Jussieu
Université Paris 7
175 rue du Chevaleret
F-75013 Paris

Abstract. After having established elementary results on the relationship between a finite complex (pseudo-)reflection group $W \subset \mathrm{GL}(V)$ and its reflection arrangement $\mathcal{A}$, we prove that the action of W on $\mathcal{A}$ is canonically related with other natural representations of W, through a 'periodic' family of representations of its braid group. We also prove that, when W is irreducible, then the squares of defining linear forms for $\mathcal{A}$ span the quadratic forms on V, which imply $|\mathcal{A}| \geq n(n+1)/2$ for $n = \dim V$, and relate the W-equivariance of the corresponding map with the period of our family.

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1. Introduction

Let V be finite-dimensional $\mathbb{C}$-vector space, $W \subset \mathrm{GL}(V)$ be a finite (pseudo-)reflection group with corresponding hyperplane arrangement $\mathcal{A}$. We assume that $\mathcal{A}$ is essential, meaning that $\bigcap \mathcal{A} = \{0\}$ and denote $n = \dim V$ the rank of W. We recall that an arrangement $\mathcal{A}$ is called irreducible if it cannot be written as $\mathcal{A}_1 \times \mathcal{A}_2$, and that W is called irreducible if it acts irreducibly on V. A basic result can be written as follows

(0) $\mathcal{A}$ is irreducible iff W is irreducible.

Steinberg showed that the exterior powers of V are irreducible. His proof is based on the encryption of irreducibility in the connectedness of certain graphs. From this approach, the following is easily deduced

(1) If W is irreducible, then it contains an *irreducible* parabolic subgroup.

Although this result is probably well-known to experts and easily checked, it does not seem to appear in print, and is a key tool for the sequel.

We then consider the permutation W-module $\mathbb{C}\mathcal{A}$. A choice of linear maps $\alpha_H \in V^*$ with kernel $H \in \mathcal{A}$ defines a linear map $\Phi : \mathbb{C}\mathcal{A} \to S^2V^*$ through $\alpha_H \mapsto \alpha_H^2$. This map can be chosen to be a morphism of W-modules when W is a Coxeter group. We prove

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(2) Φ is onto iff W is irreducible

meaning that each quadratic form on V is a linear combination of the quadratic forms α_H^2, as soon as W is irreducible. As a corollary, we get

(3) The cardinality of $\mathcal{A}$ is at least $n(n+1)/2$.

This lower bound is better than the usual $|\mathcal{A}| \geq n/2$ of [OT], cor. 6.98, and is sharp, as $|\mathcal{A}| = n(n+1)/2$ when W is a Coxeter group of type A_n.

We denote d_H the order of the (cyclic) fixer in W of $H \in \mathcal{A}$, and define the distinguished reflection $s \in W$ to be the reflection in W with $H = \mathrm{Ker}(s-1)$ and additional eigenvalue $\zeta_H = \exp(2\mathrm{i}\pi/d_H)$. We let $d : \mathcal{A} \to \mathbb{Z}$ denote $H \mapsto d_H$. We did not find the following in the standard textbooks :

(4) The data $(\mathcal{A}, d)$ determines W.

Letting B denote the braid group associated to W, we show that $\mathbb{C}\mathcal{A}$, considered as a linear representation of B, can be deformed through a path in $\mathrm{Hom}(B, \mathrm{GL}(V))$ which canonically connects $\mathbb{C}\mathcal{A}$ to other representations of W. This turns out to provide a natural generalization of the action of Weyl groups on their positive roots to arbitray reflection groups.

Finally, we prove that this path $h \mapsto R_h$ is periodic, namely that $R_{h+\kappa(W)} \simeq R_h$ for some integer $\kappa(W)$, with $\kappa(W) = 2$ when W is a Coxeter group. Moreover, $\kappa(W) = 2$ if and only if the morphism Φ above can be chosen to be a morphism of W-modules. In particular, we get

(5) If $\kappa(W) = 2$ then the W-module S^2V^* is a quotient of $\mathbb{C}\mathcal{A}$.

We emphasize the fact that the proofs presented here are elementary in the sense that, except for one of the last results, no use is made either of the Shephard-Todd classification of pseudo-reflection groups, nor of the invariants theory of these groups.

2. Reflection groups and reflection arrangements

We recall from [OT] the following basic notions about reflection groups and hyperplane arrangements. An endomorphism $s \in \mathrm{GL}(V)$ is called a (pseudo-)reflection if it has finite order and $\mathrm{Ker}(s-1)$ is an hyperplane of V. A finite subgroup W of some $\mathrm{GL}(V)$ which is generated by reflections is called a (complex) (pseudo-)reflection group. The hyperplane arrangement associated to it is the collection $\mathcal{A}$ of the reflecting hyperplanes $\mathrm{Ker}(s-1)$ for s a reflection of W. There is a natural function $d : \mathcal{A} \to \mathbb{Z}, H \mapsto d_H$ which associates to each $H \in \mathcal{A}$ the order of the subgroup of W fixing H. We let $\zeta_H = \exp(2\mathrm{i}\pi/d_H)$, and call a reflection s distinguished if its nontrivial eigenvalue is ζ_H, with $\mathrm{Ker}(s-1) = H$.

A nontrivial subgroup W_0 of W is called *parabolic* if it is the fixer of some linear subspace of V. By a fundamental result of Steinberg, this linear supspace lies inside some intersection of reflecting hyperplanes, and W_0 is also a reflection group in $\mathrm{GL}(V)$.

In general, a (central) hyperplane $\mathcal{A}$ arrangement is a finite collection of linear hyperplanes in V. When $\mathcal{A}$ originates from a reflection group W, then $\mathcal{A}$ is called a reflection arrangement. An arrangement $\mathcal{A}$ is called essential if $\bigcap \mathcal{A} = \{0\}$; for two arrangements $\mathcal{A}_1, \mathcal{A}_2$ in V_1, V_2, the arrangement $\mathcal{A}$ in $V = V_1 \times V_2$ is defined as $\{H \oplus V_2; H \in \mathcal{A}_1\} \cup \{V_1 \oplus H; H \in \mathcal{A}_2\}$; two

arrangements in V are isomorphic if they are deduced one from the other by some element of $\mathrm{GL}(V)$; an essential arrangement $\mathcal{A}$ is called irreducible if it is not isomorphic to some nontrivial $\mathcal{A}_1 \times \mathcal{A}_2$.

The following lemma shows that, when $\mathcal{A}$ is a reflection arrangement, the arrangement $\mathcal{A}$ together with the order of the reflections determines the reflection group. In particular, there is at most one reflection group with reflections of order 2 admitting a given reflection arrangement. Notice that $\mathcal{A}$ can be assumed to be essential, as the action of W on $\bigcap \mathcal{A}$ is necessarily trivial. Although basic, this fact does not appear in standard textbooks. The proof given here has been found in common with François Digne and Jean Michel.

Proposition 2.1. *Let $\mathcal{A}$ be an essential hyperplane arrangement in V.*

(1) *If $P \in \mathrm{GL}(V)$ satisfies $P(H) \subset H$ for all $H \in \mathcal{A}$, then P is semisimple.*
(2) *If $\mathcal{A}$ is a reflection arrangement associated to a complex reflection group $W \subset \mathrm{GL}(V)$, then $(\mathcal{A}, d)$ determines W.*

Proof. To prove (1), we choose linear forms $\alpha_H \in V^*$ with kernel $H \in \mathcal{A}$. Since $\mathcal{A}$ is essential, V^* is generated by the α_H, hence admits a basis made out some of them. The assumption then states that the α_H are eigenvectors for ${}^tP \in \mathrm{GL}(V^*)$, hence ${}^tP$ is semisimple and so is P. Now we prove (2), assuming that $W_1, W_2 \subset \mathrm{GL}(V)$ are two reflection groups with the same data $(\mathcal{A}, d)$. Let $H \in \mathcal{A}$ and $s_i \in W_i$ the distinguished reflection with $\mathrm{Ker}(s_i - 1) = H$. Then $x = s_1 s_2^{-1}$ fixes H and acts by 1 on V/H, hence is unipotent. The endomorphism $x \in \mathrm{GL}(V)$ clearly permutes the hyperplanes. Since $\mathcal{A}$ is finite, some power of x setwise stabilizes every $H \in \mathcal{A}$, hence is semisimple by (1). Since it is also unipotent this power of x is the identity, hence $x = \mathrm{Id}$ because x is unipotent. It follows that $s_1 = s_2$ hence $W_1 = W_2$. □

3. A consequence of Steinberg lemma

Let $W \subset \mathrm{GL}(V)$ be a reflection group and $\mathcal{A}$ the corresponding reflection arrangement. A basic fact is that the notions of irreducibility for W and $\mathcal{A}$ coincide and can be checked combinatorially on some graph. After recalling a proof of this, we notice a useful consequence.

We endow V with a W-invariant hermitian scalar product. Call $v \in V$ a *root* if it is an eigenvector of a reflection $s \in V$ such that $s.v \neq v$. For L a finite set of linearly independent roots we let V_L denote the subspace of V spanned by L, and Γ_L the graph on L connecting v_1 and v_2 if and only if v_1 and v_2 are not orthogonal. Notice that, if $s \in W$ is a reflection with root $v \in V$, the following properties hold : if $v \in V_L$ then $s(V_L) \subset V_L$, because $V_L = (\mathbb{C}v) \oplus (\mathrm{Ker}(s-1) \cap V_L)$; if $v \in V_L^{\perp}$ then $V_L \subset (\mathbb{C}v)^{\perp}$ is pointwise stabilized by s.

The following proposition is basic. We provide a proof of (1) $\Leftrightarrow$ (2) for the convenience of the reader, because of a lack of reference. (1) $\Leftrightarrow$ (3) is due to Steinberg.

Proposition 3.1. *The following are equivalent, for an essential reflection arrangement $\mathcal{A}$.*

(1) *W acts irreducibly on V.*
(2) *$\mathcal{A}$ is an irreducible hyperplane arrangement.*
(3) *V admits a basis L of roots such that Γ_L is connected.*

Proof. In the direction (2) $\Rightarrow$ (1), if $V = V_1 \oplus V_2$ with the V_i being W-stable subspaces, then we define $\mathcal{A}_i = \{H \in \mathcal{A} \mid (s_H)_{|V_i} \neq \mathrm{Id}\}$ with s_H the distinguished reflection w.r.t. $H \in \mathcal{A}$, and we have $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. In the direction (1) $\Rightarrow$ (2), we let $V = V_1 \oplus V_2$ be the decomposition of V corresponding to $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. We choose a collection of roots for $\mathcal{A}$. Let s_1, s_2 be two distinguished reflections associated to $H_1 \in \mathcal{A}_1, H_2 \in \mathcal{A}_2$, respectively, and let $H = H_1 \oplus H_2 \subset V$. Consider some reflection $s \in W$ such that $\mathrm{Ker}(s-1) \supset H$. If $\mathrm{Ker}(s-1)$ can be written as $H_0 \oplus V_2$ with H_0 some hyperplane of V_1, then $H_0 \oplus V_2 \supset H_1 \oplus H_2$ implies $H_0 \supset H_1$, hence $H_0 = H_1$ by equality of dimensions, meaning that s is some power of s_1. Similarly, if $\mathrm{Ker}(s-1)$ can be written as $V_1 \oplus H_0$ with H_0 some hyperplane of V_2, then s is a power of s_2. Considering the reflection $s_2 s_1 s_2^{-1}$, which fixes H and has reflecting hyperplane $s_2.\mathrm{Ker}(s_1 - 1)$, since $s_1 \neq s_2$ it follows that $s_2 s_1 s_2^{-1}$ is a power of s_1. Then $s_2.\mathrm{Ker}(s_1-1) = \mathrm{Ker}(s_1-1)$ hence s_1, s_2 commute and have orthogonal roots. The subspace V_1^0 spanned by all roots aring from $\mathcal{A}_1$ is thus setwise stabilized by all reflections of W, hence $V_1^0 = V$. On the other hand, the hermitian scalar product induces an isomorphism between V_1^0 and V_1^* (because $\mathcal{A}_1$, like $\mathcal{A}$, is essential), hence $V_2 \neq \{0\} \Rightarrow V_1^0 \neq V$, a contradiction.

We now prove (1) $\Leftrightarrow$ (3). Let L_0 be of maximal size among the sets L of linearly independent roots with connected Γ_L. We prove that $|L| = \dim V$ if W is irreducible. Indeed, since W is irreducible generated by reflections and $V_{L_0} \subset V$, there would otherwise exist a reflection s such that $s(V_{L_0}) \not\subset V_{L_0}$. Letting $v \in V$ be a root of s, we have $v \notin V_{L_0}$ and $v \notin (V_{L_0})^{\perp}$. This proves that $L = L_0 \sqcup \{v\}$ is made out linearly independant roots and that Γ_L is connected, since $v \notin (V_{L_0})^{\perp}$ cannot be orthogonal to all roots spanning L_0 and L_0 is already connected. From this contradiction it follows that L_0 has cardinality $\dim V$. Conversely, if V admits a basis L of roots such that Γ_L is connected, then W is irreducible, for otherwise $V = V_1 \oplus V_2$ with V_1, V_2 nontrivial orthogonal W-stable subspaces, and $L = L_1 \sqcup L_2$ with $L_i = \{x \in L \mid x \in U_i\}$. Then $\Gamma_L = \Gamma_{L_1} \sqcup \Gamma_{L_2}$, contradicting the connectedness of Γ_L. □

Corollary 3.2. *If $W \subset \mathrm{GL}(V)$ is an irreducible reflection group then it admits an* irreducible *parabolic subgroup of rank* $\dim V - 1$.

Proof. Considering a set L of linearly independent roots such that Γ_L is connected, as given by the proposition, there exists $L_0 \subset L$ with $L = L_0 \sqcup \{v\}$ such that Γ_{L_0} is still connected. Then V_{L_0} has dimension $\dim V - 1$, and its orthogonal is spanned by some $v' \in V$. Letting W_0 denote the parabolic subgroup fixing v', it has rank $\dim V - 1$, admits for roots all elements of L_0, hence is irreducible since Γ_{L_0} is connected. □

4. Quadratic forms on V

Let $\mathcal{A}$ be an essential hyperplane arrangement in V. The integer $n = \dim V$ is the *rank* $\mathrm{rk}\,\mathcal{A}$ of $\mathcal{A}$. For each $H \in \mathcal{A}$ we let $\alpha_H \in V^*$ denote

some linear form with kernel H. For a field $\mathbb{k}$, we let $\mathbb{k}\mathcal{A}$ denote a vector space with basis $v_H, H \in \mathcal{A}$, and define a linear map $\Phi : \mathbb{C}\mathcal{A} \to S^2V^*$ by $\Phi(v_H) = \alpha_H^2$.

For Φ to be onto, it is nessary that $\mathcal{A}$ is irreducible. Indeed, if $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ corresponds to some direct sum decomposition $V = V_1 \oplus V_2$, then choosing two nonzero linear forms $\varphi_i \in V_i^*$ defines a quadratic form $\varphi_1\varphi_2 \in S^2V^*$ which does not belong to $\operatorname{Im}\Phi$. This condition is also sufficient in rank 2.

Proposition 4.1. *If $\mathcal{A}$ is essential of rank 2, then Φ is onto if and only if $\mathcal{A}$ is irreducible.*

Proof. Since $\mathcal{A}$ is essential, $\mathcal{A}$ contains at least two hyperplanes H_1, H_2. We denote $\alpha_i = \alpha_{H_i}$ the corresponding (linearly independant) linear forms. If $\mathcal{A} = \{H_1, H_2\}$, then $\mathcal{A}$ is obviously reducible, so we may assume that $\mathcal{A}$ contains at least another hyperplane. Let β denote the corresponding linear form. It can be written as $\beta = \lambda_1\alpha_1 + \lambda_2\alpha_2$ with $\lambda_1 \neq 0$, $\lambda_2 \neq 0$. Since $\beta^2 = \lambda_1^2\alpha_1^2 + 2\lambda_1\lambda_2\alpha_1\alpha_2 + \lambda_2^2\alpha_2^2$ and $\alpha_1^2, \alpha_2^2, \beta^2 \in \operatorname{Im}\Phi$ we get $\alpha_1\alpha_2 \in \operatorname{Im}\Phi$. Since $\alpha_1^2, \alpha_2^2 \in \operatorname{Im}\Phi$ and α_1, α_2 are linearly independent it follows that $\operatorname{Im}\Phi = S^2V^*$. □

This condition is not sufficient in rank 3, as shows the following example. Consider in $\mathbb{C}^3$ the central arrangement of polynomial $xyz(x-y)(y-z)$. The morphism Φ is obviously not surjective, as $\dim \mathbb{C}\mathcal{A} = 5$ and $\dim S^2V^* = 6$. However, $\mathcal{A}$ is irreducible, because its Poincaré polynomial is $P_{\mathcal{A}}(t) = (1 + t)(1 + 4t + 4t^2)$, which is not divisible by $(1 + t)^2$ — recall from [OT] that $P_{\mathcal{A}_1\times\mathcal{A}_2} = P_{\mathcal{A}_1}P_{\mathcal{A}_2}$ and that $P_{\mathcal{A}}(t)$ is divisible by $1+t$ whenever $\mathcal{A}$ is central.

It is however sufficient when $\mathcal{A}$ is a *reflection arrangement.*

Theorem 4.2. *Let $\mathcal{A}$ be a (essential) reflection arrangement. Then Φ is surjective if and only if $\mathcal{A}$ is irreducible.*

Proof. We assume that $\mathcal{A}$ is irreducible, and prove that Φ is surjective by induction on $\operatorname{rk}\mathcal{A}$. If $\operatorname{rk}\mathcal{A} \leq 2$, this is a consequence of the above proposition, so we can assume $\operatorname{rk}\mathcal{A} \geq 3$. We denote W the corresponding (pseudo-)reflection group, and endow V with a W-invariant hermitian scalar product. By corollary 3.2 there exists an irreducible maximal parabolic subgroup $W_0 \subset W$, defined by $W_0 = \{w \in W \mid w.v = v\}$ for some $v \in V \setminus \{0\}$. We let $H_0 = (\mathbb{C}v)^{\perp}$. By Steinberg theorem W_0 is a reflection group, whose (pseudo-)reflections are the reflections of W contained in W_0. Let $\mathcal{A}_0 \subset \mathcal{A}$ denote the arrangement in V corresponding to W_0. Since $v \in H$ for all $H \in \mathcal{A}_0$, by the induction hypothesis we have $Q \subset S^2H_0^*$, where $Q = \operatorname{Im}\Phi$ and $S^2H_0^* \subset S^2V^*$ is induced by $H^* \subset V^*$, letting $\gamma \in H_0^*$ act on $H_0^{\perp}$ by 0. Let $\alpha \in V^*\setminus\{0\}$ such that $H_0 = \operatorname{Ker}\alpha$. We have $S^2V^* = S^2H_0^* \oplus \alpha H_0^* \oplus \mathbb{C}\alpha^2$. Since $\mathcal{A}$ is irreducible, there exists $H \in \mathcal{A}$ such that $\alpha_H \notin \mathbb{C}\alpha$ and $\alpha_H \notin S^2H_0^*$. Such a linear form can be written $\lambda(\alpha + \beta)$ with $\lambda \in \mathbb{C} \setminus \{0\}$ and $\beta \in S^2H_0^* \setminus \{0\}$. Then $(\alpha + \beta)^2 \in Q$ and $\beta^2 \in Q$, so we have $\alpha^2 + 2\alpha\beta \in Q$. We make W act on V^* by $w.\gamma(x) = \gamma(w^{-1}.x)$, for $x \in V$, $\gamma \in V^*$. Of course this action can be restricted to a W_0-action on $H_0^* \subset V^*$. Then $w.(\alpha + \beta) \in Q$ for all $w \in W$, and since $w.\alpha = \alpha$ whenever $w \in W_0$, we get $\alpha^2 + 2\alpha(w.\beta) \in Q$ for all $w \in W_0$. Consider now the subspace U of H^* spanned by the $w_1.\beta - w_2.\beta$ for $w_1, w_2 \in W_0$. It is a W_0-stable

subspace of H_0^*. Recall that H_0, hence H_0^*, is irreducible under the action of W_0. If $U = \{0\}$ then $w.\beta = \beta$ for all $w \in W_0$, hence $H_0 = \mathbb{C}\beta$ and $\dim V = 2$, which has been excluded. Thus $U \neq \{0\}$ hence $U = H_0^*$. By $2\alpha(w_1.\beta - w_2.\beta) = (\alpha^2 + 2\alpha(w_1.\beta)) - (\alpha^2 + 2\alpha(w_2.\beta))$ we thus get $\alpha H_0^* \subset Q$. Then $(\alpha+\beta)^2 \in \alpha^2 + \alpha H_0^* + S^2 H_0^* \subset \alpha^2 + Q$ implies $\alpha^2 \in Q$. It follows that $Q \supset S^2 V^*$ which concludes the proof. □

Corollary 4.3. *If $\mathcal{A}$ is an irreducible reflection arrangement of rank n, then $|\mathcal{A}| \geq n(n+1)/2$.*

Notice that the above lower bound is sharp, as it is reached for Coxeter type A_n.

When $\mathcal{A}$ is a reflection arrangement with corresponding reflection group W, both $\mathbb{C}\mathcal{A}$ and S^2V^* can be endowed by natural W-actions, where the action on $\mathbb{C}\mathcal{A}$ is defined by $w.v_H = v_{w(H)}$. It is thus natural to ask whether the linear forms α_H can be chosen such that Φ is a morphism of W-modules.

Proposition 4.4. *If $\mathcal{A}$ is a complexified* real *reflection arrangement (in particular W is a finite Coxeter group), then the linear forms α_H can be chosen such that Φ is a morphism of W-modules.*

Proof. We choose a W-invariant scalar product on the original real form V_0 of V and extend it to a W-invariant hermitian scalar product on V. For every $H \in \mathcal{A}$ we choose $x_H \in V_0$ orthogonal to H with norm 1, and define $\alpha_H : y \mapsto (x|y)$, our convention on hermitian scalar products being that they are linear on the right. Then, for any $w \in W$, $w.x_H \in V_0$ is orthogonal to $w(H)$ of norm 1, hence $w.x_H = \pm x_{w(H)}$. Since $w.\alpha_H$ maps y to $(w.x_H|y)$ we have $(w.\alpha_H)^2 = \alpha_{w(H)}^2$, which shows that Φ is a morphism of W-modules. □

When W is not a Coxeter group, the W-modules $\mathbb{C}\mathcal{A}$ and S^2V^* are generally unrelated. However, this property is not a characterization of Coxeter groups, as there is at least one example of a (non-Coxeter) complex reflection group for which Φ can be a morphism of W-module. This is the group labelled G_{12} in the Shephard-Todd classification. Notice that, in such a case, one must have $\sum \alpha_H^2 = 0$, otherwise this sum would provide a copy of the trivial representation inside S^2V^*, forcing W to be a real reflection group.

We briefly describe this example. The group G_{12} can be described in $\mathrm{GL}_2(\mathbb{C})$ by 3 generators a, b, c of order 2, satisfying the relation $abca = bcab = cabc$. We choose the following model :

$$a = \begin{pmatrix} 1 & 1+\sqrt{-2} \\ 0 & -1 \end{pmatrix} b = \begin{pmatrix} -1 & 0 \\ 1-\sqrt{-2} & 1 \end{pmatrix} c = \begin{pmatrix} \sqrt{-2} & -1+\sqrt{-2} \\ -1-\sqrt{-2} & -\sqrt{-2} \end{pmatrix}$$

We define a collection of vectors $e_H \in V$, such that $w.e_H = \pm e_{w(H)}$. Letting $\alpha_H : x \mapsto (e_H|x)$, the associated $\Phi : \mathbb{C}\mathcal{A} \to S^2V^*$ is then a morphism of W-modules. A W-invariant hermitian scalar product is given on this matrix model by $(X|Y) = {}^t\bar{X}AY$ with

$$A = \begin{pmatrix} 2 & 1+\sqrt{-2} \\ 1-\sqrt{-2} & 2 \end{pmatrix}$$

We choose for e_H the 12 following vectors, which are fixed by the corresponding reflection s.

s	$babab$	a	b
e_H	$(1+\sqrt{-2},-2)$	$(1,0)$	$(0,1)$
s	$ababa$	bcb	c
e_H	$(-2,1-\sqrt{-2})$	$(1,\sqrt{-2})$	$(1,-1)$
s	$acaca$	cbc	aba
e_H	$(1-\sqrt{-2},1+\sqrt{-2})$	$(-1+\sqrt{-2},-\sqrt{-2})$	$(-1-\sqrt{-2},1)$
s	bab	cac	aca
e_H	$(-1,1-\sqrt{-2})$	$(-\sqrt{-2},1+\sqrt{-2})$	$(-\sqrt{-2},1)$

It can be checked that the reflections a,b,c act on these vectors by monomial matrices, with nonzero entries in $\{\pm 1\}$ (hence factors through the hyperoctahedral group of rank 12). On this example, S^2V^* is a selfdual W-module.

We make the following remark.

Proposition 4.5. *For Φ to be a morphism of W-modules it is necessary that $\kappa(W)\leq 2$, where*

$$\kappa(W)=\min\{n\in\mathbb{Z}_{>0}\ |\forall w\in W\ \forall H\in\mathcal{A}\ \ w.\alpha_H=\zeta\alpha_H\Rightarrow\zeta^n=1\}$$

Using the Shephard-Todd classification, we will show in section 6 that this condition is actually sufficient when W is irreducible.

5. A path between representations

In this section we define a natural connection between the action of W on $\mathbb{C}\mathcal{A}$ and more surprising representations of W. For this we need to introduce the space $X=V\setminus\bigcup\mathcal{A}$ of regular vectors, on which W acts freely, and its quotient (orbit) space X/W. We choose a base point $\underline{z}\in X$. The fundamental groups $B=\pi_1(X/W)$ and $P=\pi_1(X)$ are known as the braid group and pure braid group associated to W, respectively. There is a natural morphism $\pi:B\to W$ with kernel P. We first construct a deformation of $W\to\mathrm{GL}(\mathbb{C}\mathcal{A})$ as a linear representation of the braid group. This deformation should not be confused with the one described in [Ma07] when W is a 2-reflection group.

5.1. A representation of the braid group.

To each $H\in\mathcal{A}$ is canonically associated a differential form $\omega_H=\frac{\mathrm{d}\,\alpha_H}{\alpha_H}$, using some arbitrary linear form α_H with kernel α_H. We introduce idempotents $p_H\in\mathrm{End}(\mathbb{C}\mathcal{A})$ defined by $p_{H_1}.v_{H_2}=v_{H_2}$ if $H_1=H_2$, $p_{H_1}.v_{H_2}=0$ otherwise. Choosing $h\in\mathbb{C}$, the 1-form

$$\omega=h\sum_{H\in\mathcal{A}}p_H\omega_H\in\Omega^1(X)\otimes\mathfrak{gl}(\mathbb{C}\mathcal{A})$$

satisfies $\omega\wedge\omega=0$, hence defines a flat connection on the trivial vector bundle $X\times\mathbb{C}\mathcal{A}\to X$, which is clearly W-equivariant for the diagonal action on $X\times\mathbb{C}\mathcal{A}$. Dividing out by W, the corresponding flat bundle over X/W thus defines by monodromy a linear representation of B in $\mathbb{C}\mathcal{A}$. Letting γ denote a representative loop of $\sigma\in B=\pi_1(X/W)$, we can lift it to a path $\tilde{\gamma}$ in X with endpoints $\underline{z}$ and $\pi(\sigma).\underline{z}$, where $\underline{z}$ is the chosen basepoint

in X. The 1-forms $\tilde{\gamma}^*\omega_H$ can be written as $\gamma_H(t)\,\mathrm{d}\,t$ for some function γ_H on $[0,1]$, and the differential equation $\mathrm{d}\,f = (\gamma^*\omega)f$ to consider is then $f'(t) = h(\sum_{H\in\mathcal{A}}\gamma_H(t)p_H)f(t)$, with $f(0) = \mathrm{Id} \in \mathrm{End}(\mathbb{C}\mathcal{A})$. Since the p_H commute one to the other, the solution is easy to compute :

$$f(t) = \prod_{H\in\mathcal{A}} \exp\left(hp_H \int_0^t \gamma_H(u)\,\mathrm{d}\,u\right)$$

and the monodromy representation is given by

$$\sigma \mapsto R_h(\sigma) = \pi(\sigma) \prod_{H\in\mathcal{A}} \exp(hp_H \int_\gamma \omega_H)$$

where we identified $w \in W$ with $R_0(w) \in \mathrm{End}(\mathbb{C}\mathcal{A})$. In particular, the image of P is commutative. More precisely, if γ_0 is a loop in X around a single hyperplane H, the class $[\gamma_0] \in P$ is mapped to $\exp(2\mathrm{i}\pi hp_H)$. Since P is generated by such classes, it follows that $R_n(P) = \{\mathrm{Id}\}$ hence R_n factors through a representation of W whenever $n \in \mathbb{Z}$.

We recall that B is generated by so-called braided reflections ('generators-of-the-monodromy' in [BMR]), which are defined as follows. For a distinguished reflection $s \in W$, an element $\sigma \in B$ with $\pi(\sigma) = s$ is called a braided reflection if it admits as representative a path γ from $\underline{z}$ to $s.\underline{z}$ which is a composite $(s.\gamma_0)^{-1} * \gamma_1 * \gamma_0$ of paths with the following properties. Here $\gamma_0 : \underline{z} \rightsquigarrow \underline{z}_0$, $\gamma_1 : \underline{z_0} \rightsquigarrow s.\underline{z_0}$ and $(s.\gamma_0)^{-1} : s.\underline{z}_0 \rightsquigarrow s.\underline{z}$ is the reverse path of $s.\gamma_0$, and $\gamma_1(t) = \varepsilon \exp(2\mathrm{i}\pi t/d_H)\underline{z_0}^- + \underline{z_0}^+$ where $\underline{z_0}^+$ and $\underline{z_0}^-$ are the orthogonal projection on H and $H^\perp$, respectively, for $\varepsilon > 0$ small enough and $\underline{z_0}$ sufficiently close to H so that the homotopy class of this path does not vary when ε decreases and $\underline{z_0}^+ \notin H'$ for $H' \in \mathcal{A} \setminus \{H\}$.

Note that $\int_{s.\gamma_0}\omega_{H'} = \int_{\gamma_0}\omega_{s(H')}$ for all $H' \in \mathcal{A}$, hence $\int_\gamma \omega_H = \int_{\gamma_1}\omega_H = (2\mathrm{i}\pi)/d_H$. In particular, for such a braided reflection σ we get

$$R_h(\sigma).v_H = \pi(\sigma)\exp(hp_H \int_\gamma \omega_H)v_H = \exp(2\mathrm{i}\pi h/d_H)v_H.$$

Moreover, if H and H' have orthogonal roots, then again $\int_\gamma \omega_{H'} = \int_{\gamma_1}\omega_{H'}$. But in this case $\alpha_{H'}(\gamma_1(t))$ is constant hence $\int_\gamma \omega_{H'} = 0$. An immediate consequence of this is that we can restrict ourselves to irreducible groups, namely

Proposition 5.1. *If $W = W_1 \times \cdots \times W_r$ is a decomposition of W in irreducible components, with corresponding decompositions $B = B_1 \times \cdots \times B_k$ and $\mathcal{A} = \mathcal{A}^1 \times \dots \mathcal{A}^r$, then $R_h = R_h^{(1)} \times \cdots \times R_h^{(r)}$ with $R_h^{(k)} : W_k \to \mathrm{GL}(\mathbb{C}\mathcal{A}^k)$.*

From the formulas above follows that, under the action of R_h, $\mathbb{C}\mathcal{A}$ is the direct sum of the stable subspaces $\mathbb{C}\mathcal{A}_k$, where $\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_r$ is the decomposition of $\mathcal{A}$ in orbits under the action of W. We let $R_h^k : B \to \mathrm{GL}(\mathbb{C}\mathcal{A}_k)$, so that $R_h = R_h^1 \oplus \cdots \oplus R_h^r$.

Proposition 5.2. *If $h \notin \mathbb{Z}$, then R_h^k is irreducible for each $1 \leq k \leq r$.*

Proof. For each $H \in \mathcal{A}_k$ we choose a loop γ_H based at $\underline{z}$ around the hyperplane H, We have $\int_{\gamma_H}\omega_H = 2\mathrm{i}\pi$ and $\int_{\gamma_H}\omega_{H'} = 0$ for $H \neq H'$. Letting Q_H

denote the class of γ_H in $P = \pi_1(X, \underline{z})$ we thus have $R_h^k(Q_H) = \exp(2\mathrm{i}\pi h p_H)$, hence $R_h^k(Q_H) - \mathrm{Id}$ is a nonzero multiple of p_H if $h \notin \mathbb{Z}$. It follows that the elements $R_h^k(Q_H)$ generate the commutative algebra of diagonal matrices in $\mathrm{End}(\mathbb{C}\mathcal{A}_k)$. Let $\mathcal{G}_k$ be the oriented graph on the $v_H, H \in \mathcal{A}_k$ with an edge (v_{H_1}, v_{H_2}) if there exists $x \in B$ such that the matrix $R_h^k(x)$ has nonzero entry at (v_{H_1}, v_{H_2}). If $\mathcal{G}_k$ is connected, then R_h^k is irreducible (see e.g. [Ma04] prop. 3 cor. 2). Choosing for each distinguished reflection $s \in W$ a braided reflection σ, $R_h^k(\sigma)$ has nonzero entries in $(v_H, v_{s(H)})$ and $(v_{s(H)}, v_H)$ for each $H \in \mathcal{A}$. Since $\mathcal{A}_k$ is an orbit under W and W is generated by distinguished reflections, it follows that $\mathcal{G}_k$ is connected, concluding the proof. □

Since R_h factors through W when $h \in \mathbb{Z}$, this has the following consequence.

Corollary 5.3. *For all $h \in \mathbb{C}$, the representation R_h of B is semisimple.*

We choose a collection of roots $e_H, H \in \mathcal{A}$. Notice that, for $w \in W$, $w(H) = H$ implies $w.e_H = e^{\mathrm{i}\theta} e_H$ for some $\theta \in \mathbb{R}$.

Lemma 5.4. *If $\gamma : \underline{z} \rightsquigarrow w.\underline{z}$ is a path in X with $w \in W$ such that $w.e_H = e^{\mathrm{i}\theta} e_H$, then $\int_\gamma \omega \in \mathrm{i}\theta + 2\mathrm{i}\pi\mathbb{Z}$.*

Proof. We can assume $-\pi < \theta \leq \pi$. Since $\int_\gamma \omega_H$ is independent of the choice of α_H, we can choose $\alpha_H : x \mapsto (e_H|x)$ with $(e_H|e_H) = 1$. We have $\alpha_H(w.x) = e^{\mathrm{i}\theta}\alpha(x)$. We write $\gamma(t) = \gamma_H(t) + \gamma_0(t)e_H$ with $\gamma_0 : [0,1] \to \mathbb{C}$ and $\gamma_H : [0,1] \to H$. Then $\alpha_H(\gamma(t)) = \gamma_0(t)$ and $\int_\gamma \omega_H = \int_{\gamma_0} \frac{\mathrm{d}z}{z}$. Letting $x = \alpha_H(\underline{z}) \in \mathbb{C}^\times$, we have $\gamma_0 : x \rightsquigarrow e^{\mathrm{i}\theta}x$. If $\gamma_1 : x \rightsquigarrow e^{\mathrm{i}\theta}x$ is an arbitrary path in $\mathbb{C}^\times$, then $\gamma_0 * \gamma_1^{-1}$ is a loop in $\mathbb{C}^\times$, hence $\int_{\gamma_0} \frac{\mathrm{d}z}{z} - \int_{\gamma_1} \frac{\mathrm{d}z}{z}$ is a multiple of $2\mathrm{i}\pi$. If $e^{\mathrm{i}\theta} = 1$ this concludes the proof. If $e^{\mathrm{i}\theta} = -1$ we consider $\gamma_1(t) = xe^{\mathrm{i}\pi t}$, for which $\int_{\gamma_1} \frac{\mathrm{d}z}{z} = \mathrm{i}\pi$. If $e^{\mathrm{i}\theta} = \zeta \notin \{1, -1\}$ we consider $\gamma_1(t) = (1-t)x + te^{\mathrm{i}\theta}x$ and $\int_{\gamma_1} \frac{\mathrm{d}z}{z} = \log(1 + (e^{\mathrm{i}\theta} - 1)t)\Big|_0^1$ where log denotes the natural determination of the logarithm over $\mathbb{C} \setminus \mathbb{R}^-$. It follows that $\int_{\gamma_1} \frac{\mathrm{d}z}{z} = \log e^{\mathrm{i}\theta} = \mathrm{i}\theta$, and the conclusion follows. □

We recall from section 4 the definition of $\kappa(W)$.

$$\kappa = \kappa(W) = \min\{n \in \mathbb{Z}_{>0} \ |\forall w \in W \ \forall H \in \mathcal{A} \ \ w.e_H = \zeta e_H \Rightarrow \zeta^n = 1\}$$

Theorem 5.5. *For all $h \in \mathbb{C}$, $R_{h+\kappa}$ is isomorphic to R_h. Moreover, κ is the smallest positive real number such that $R_\kappa \simeq R_0$.*

Proof. Recall from corollary 5.3 that, for all $h \in \mathbb{C}$, R_h is semisimple. Letting χ_h denote the character of R_h on B, it is thus sufficient to prove $\chi_h = \chi_{h+\kappa}$ for all $h \in \mathbb{C}$ in order to get $R_{h+\kappa} \simeq R_h$. Let $g \in B$ with $w = \pi(g)$, and $\gamma : \underline{z} \rightsquigarrow w.\underline{z}$ a representing path. By the explicit formulas above, we have

$$\chi_h(g) = \sum_{w(H)=H} \exp(h \int_\gamma \omega_H)$$

and $R_{h+\kappa} \simeq R_h$ follows by lemma 5.4. We now show that κ is minimal with this property. Assuming otherwise, we let $0 < h < \kappa$ such that $\chi_h = \chi_0$. By definition of κ there exists $w \in W$, $H \in \mathcal{A}$ such that $w.e_H = e^{\mathrm{i}\theta} e_H$ with

$e^{\mathrm{i}\theta h} \neq 1$. Letting $g \in B$ with $\pi(g) = w$ and $\gamma : \underline{z} \rightsquigarrow w.\underline{z}$ a representing path, we have $\int_\gamma \omega_H \in \mathrm{i}\theta + 2\mathrm{i}\pi\mathbb{Z}$, hence $\exp(h \int_\gamma \omega_H) \neq 1$. It follows that $|\chi_h(g)| < \chi_0(g)$ hence a contradiction. □

Proposition 5.6. *For any $H \in \mathcal{A}$ and $h \in \mathbb{C}$, if σ is a braided reflection around H, then $R_h(\sigma)$ is conjugated to $R_0(\sigma)\exp(h(2\mathrm{i}\pi/d_H)p_H)$.*

Proof. Let σ be a braided reflection with corresponding paths $\gamma, \gamma_0, \gamma_1$ as above. Since γ_0 and $s.\gamma_0$ represent the same path in X/W, $R_h(\sigma)$ is conjugated to the monodromy along the loop γ_1 in X/W, so that we can assume $\underline{z} = \underline{z_0}$, $\gamma = \gamma_1$. In view of the formulas above, we thus only need to show that $\int_{\gamma_1} \omega_{H'} = 0$ for $H' \neq H$. This can be done by direct computation, as $\alpha_{H'}(\gamma_1(t)) = \varepsilon \exp(2\mathrm{i}\pi t/d_H)\alpha_{H'}(\underline{z_0}^-) + \alpha_{H'}(\underline{z_0}^+)$ with $\alpha_{H'}(\underline{z_0}^-) \neq 0$, and $\int_{\gamma_1} \omega_{H'}$ is constant when $\varepsilon \to 0$. Since $\int_{\gamma_1} \omega_{H'} \to 0$ when $\varepsilon \to 0$ we get $\int_\gamma \omega_{H'} = 0$ and the conclusion. □

5.2. **New representations of W.** When $n \in \mathbb{Z}$, the representation R_n of B factorizes through W. In case W is irreducible, the action of the center is easy to describe.

Lemma 5.7. *If $w \in W$ acts by $\lambda \in \mathbb{C}^\times$ on V, then $R_n(w) = \lambda^n \mathrm{Id}$ if $n \in \mathbb{Z}$. More generally, if there exists $v \in X$ such that $w.v = \lambda v$ for some $\lambda \in \mathbb{C}^\times$, then $R_n(w)$ is conjugated to $\lambda^n R_0(w)$*

Proof. We first assume that w acts on V by λ. We can write $\lambda = \exp(\mathrm{i}\theta)$ with $0 < \theta \leq 2\pi$. We consider the loop $\gamma(t) = e^{\mathrm{i}\theta t}\underline{z}$ in X/W, whose image in W is w. By direct calculation we have $\int_\gamma \omega_H = \mathrm{i}\theta$ for all $H \in \mathcal{A}$ and the conclusion follows from the general formula for R_1. Now assume $w.v = \lambda v$ for some $\lambda = \exp(\mathrm{i}\theta)$ with $0 < \theta \leq 2\mathrm{i}\pi$. Up to conjugation, we can assume $v = \underline{z}$, the loop $\gamma(t) = e^{\mathrm{i}\theta t}\underline{z}$ in X/W has image w in W and we conclude as before. □

More involved tools prove the following.

Proposition 5.8. *If W_0 is a parabolic subgroup of W with hyperplane arrangement $\mathcal{A}$ and $n \in \mathbb{Z}$, then the restriction of R_n to W_0 is isomorphic to the direct sum of the representation R_n of W_0 and the permutation representation of W_0 on $\mathbb{C}(\mathcal{A} \setminus \mathcal{A}_0)$.*

Proof. We let R_h^0 denote the representation R_h for W_0 acting on $\mathbb{C}\mathcal{A}_0$, and S_h the direct sum of R_h^0 and the permutation representation of W_0 on $\mathcal{A}\setminus\mathcal{A}_0$. We can embed the braid group B_0 of W_0 inside B such that, as representations over $\mathbb{C}[[h]]$, the restriction to B_0 of R_h is isomorphic to S_h (see [Ma07], theorem 2.9). In particular, for all $g \in B_0$, the traces of $R_h(g)$ and $S_h(g)$ are equal, as formal series in h. Since these traces are holomorphic functions in h, it follows that they are equal for all $h \in \mathbb{C}$. This means that the semisimple representations of B_0 associated to the restriction of R_h and to S_h are isomorphic. Since the restriction of R_n and S_n are semisimple for all $n \in \mathbb{Z}$ the conclusion follows.

□

The determination of the action of the center enables us to prove that, contrary to R_0, R_1 is faithful in general.

Proposition 5.9.
(1) *R_0 has kernel $Z(W)$.*
(2) *R_1 is faithful on W.*
(3) $\operatorname{Ker} R_n = \{w \in Z(W) \mid w^n = 1\}$

Proof. Without loss of generality (because of proposition 5.1) we may assume that W is irreducible. Obviously (3) $\Rightarrow$ (2). Although (1) is also a special case of (3), we prove it separately. If $|\mathcal{A}| = 1$ the statement is obvious, so we assume $|\mathcal{A}| \geq 2$. Clearly $Z(W) \subset \operatorname{Ker} R_0$, as $\operatorname{Ker}(wgw^{-1} - 1) = w.\operatorname{Ker}(g - 1)$ for all $g, w \in W$. Let $w \in W$ such that $R_0(w) = \mathrm{Id}$, that is $w(H) = H$ for all $H \in \mathcal{A}$. Let $s \in W$ be a distinguished reflection with reflection hyperplane H. Then wsw^{-1} is a reflection with $\operatorname{Ker}(wsw^{-1} - 1) = H$ which has the same nontrivial eigenvalue as s, hence $wsw^{-1} = s$. It follows that w commutes to all distinguished reflections of W, hence $w \in Z(W)$ since W is generated by such elements.

We now prove (3). Let $w \in \operatorname{Ker} R_n$. Since $R_1(w) = R_0(w)D$ for some diagonal matrix D, the nonzero entries of $R_n(w)$ determine the permutation matrix $R_0(w)$, hence $w \in Z(W)$. Since W is irreducible, w acts on V by some scalar $\lambda \in \mathbb{C}^\times$, hence $R_n(w) = \lambda^n = 1$ by lemma 5.7, hence $w^n = 1$. The converse inclusion is obvious by lemma 5.7. $\square$

Corollary 5.10. *The exponent of $Z(W)$ divides $\kappa(W)$. If W is irreducible then $|Z(W)|$ divides $\kappa(W)$.*

Proof. By the proposition, the period of the sequence $\operatorname{Ker} R_n$ is the exponent of $Z(W)$. Since $\operatorname{Ker} R_n$ is $\kappa(W)$-periodic the conlusion follows. If W is irreducible then $Z(W)$ is cyclic hence its order equals its exponent. $\square$

In the proof of theorem 5.5, we computed the character χ_n of R_n. We recall the result here :

Proposition 5.11. *For any $w \in W$ and $n \in \mathbb{Z}$ we have*
$$\chi_n(w) = \sum_{w.e_H = \zeta e_H} \zeta^n$$

If $\tilde{K} = \mathbb{Q}(\zeta_d)$ is a cyclotomic field containing all eigenvalues of $R_1(W)$, then letting $c_n \in \operatorname{Gal}(\tilde{K}|\mathbb{Q})$ for $n \wedge d = 1$ be defined by $c_n(\zeta_d) = \zeta_d^n$ we get from this proposition that $\chi_n = c_n \circ \chi_1$ for all n prime to d.

As an illustration of this section, we do the example of W of type G_4 generated by
$$s = \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix} \quad t = \frac{1}{3}\begin{pmatrix} 1+2j & j-1 \\ 2j-2 & j+2 \end{pmatrix}.$$
It is a reflection group of order 24, with two generators s, t of order 3 satisfying $sts = tst$, and center of order 2. It admits 3 one-dimensional (irreducible) representations $S_\alpha : s, t \mapsto \alpha$, 3 two-dimensional representations A_α with $\operatorname{tr} A_\alpha(s) = -\alpha$ for $\alpha \in \{1, j, j^2\}$ with $j = \exp(2i\pi/3)$ and a 3-dimensional one that we denote U. The reflection representation is A_{j^2}, and $\kappa(W) = 6$. From the character table of W one gets
$$\begin{array}{lllllllll} R_0 & = & S_1 + U & R_1 & = & A_1 + A_{j^2} & R_2 & = & S_{j^2} + U \\ R_3 & = & A_j + A_{j^2} & R_4 & = & S_j + U & R_5 & = & A_1 + A_{j^2} \end{array}$$

5.3. **The case of Coxeter groups.** If W is a Coxeter group, we get a simpler form of this representation. Recall that, in this case, $\mathcal{A}$ is the complexification of some real arrangement $\mathcal{A}_0$ in V_0, where V_0 is a real form of V ; moreover, choosing some connected component $\mathcal{C}$ of $V_0 \setminus \bigcup \mathcal{A}_0$, called a Weyl chamber, determines n hyperplanes $H_1, \dots, H_n$ called the *walls* of $\mathcal{C}$, and the corresponding n reflections $s_1, \dots, s_n$ are called the simple reflections associated to $\mathcal{C}$. If $\underline{z} \in \mathcal{C}$, there is also a special set of generators for B, namely the braided reflections σ_i around H_i such that γ_0 is a straight (real) segment orthogonal to H_i. These are called the Artin generators of B (associated to a choice of Weyl chamber).

Proposition 5.12. *If W is a Coxeter group with simple reflections $s_1, \dots, s_n$, then $\sigma_i \mapsto R_0(s_i)\exp(\mathrm{i}\pi h p_{H_i})$ defines a representation of B which is equivalent to R_h. In particular, R_1 is equivalent to a representation of W on $\mathbb{C}\mathcal{A}$ for which $s_i.v_H = v_{s(H)}$ is $H \neq H_i$, $s_i.v_{H_i} = -v_{H_i}$, and R_{h+2} is equivalent to R_h for any $h \in \mathbb{C}$, while $R_1 \not\simeq R_0$.*

Proof. We introduce the Weyl chamber $\mathcal{C} \subset V_0$ with respect to the simple reflections $s_1, \dots, s_n$, with walls $H_i = \mathrm{Ker}(s_i - 1)$, $1 \leq i \leq n$. Up to conjugacy the base point $\underline{z}$ can be chosen inside the Weyl chamber, and we define roots $e_H \in V_0$ of norm 1 such that $\mathbb{C}e_H = \mathrm{Ker}(s-1)^{\perp}$ and $(e_H|\underline{z}) > 0$ for $\underline{z} \in \mathcal{C}$. We choose for α_H the linear form $x \mapsto (e_H|x)$. Let us denote $\log^+$ the complex logarithm on $\mathbb{C} \setminus \mathrm{i}\mathbb{R}_-^{\times}$, and define

$$D_h = \prod_{H \in \mathcal{A}} \exp(\mathrm{i}\pi p_H \log^+(e_H|\underline{z}))$$

We consider a simple reflection s_i around a wall H_i. Then the path γ representating σ_i can be chosen with ε small enough so that $(e_H|\gamma(t))$ has positive real part for each $t \in [0,1]$ and $H \neq H_i$. It follows that $t \mapsto \log^+(e_H|\gamma(t))$ has differential $\gamma^*\omega_H$ and $R_h(\sigma_i)$ equals

$$R_0(s_i) \prod_{H \in \mathcal{A}} \exp(hp_H \int_{\gamma} \omega_H) = R_0(s_i) \prod_{H \in \mathcal{A}} \exp\left(hp_H(\log^+(e_H|s_i.\underline{z}) - \log^+(e_H|\underline{z}))\right)$$

(see [Ma07], lemma 7.10). Moreover, $(e_H|s_i.\underline{z}) = (s_i.e_H|\underline{z}) = (e_{s_i(H)}|\underline{z})$ if $H \neq H_i$ (see e.g. [Ma07], lemma 7.9) and $(e_{H_i}|s_i.\underline{z}) = -(e_{H_i}|\underline{z})$. It follows that

$$R_h(\sigma_i) = s_i \exp(\mathrm{i}\pi h p_{H_i}) \prod_{H \in \mathcal{A} \setminus \{H_i\}} \exp\left(hp_H(\log^+(e_{s_0(H)}|\underline{z}) - \log^+(e_H|\underline{z}))\right)$$

namely

$$R_h(\sigma_i) = D_h s_i \exp(\mathrm{i}\pi h p_{H_i}) D_h^{-1}$$

for all $i \in [1,n]$, which concludes the proof. $R_1 \not\simeq R_0$ because $\mathrm{tr}\, R_1(s_1) = \mathrm{tr}\, R_0(s_1) - 1$. $\square$

The representation of W described in this proposition for $h = 1$ is natural in the realm of root systems. Indeed, if a set $\mathcal{P}$ of roots for $\mathcal{A}_0$ is chosen, such that $\mathcal{P}$ satisfies the axioms $(SR)_I$ and $(SR)_{II}$ of a root system (see [Bo]), and $\mathcal{P}$ is subdivided in positive and negative roots $\mathcal{P}^+, \mathcal{P}^-$ according to the chosen Weyl chamber, where $\mathcal{P}^+ = \{e_H, H \in \mathcal{A}\}$, then the representation described here is isomorphic to one on $\mathbb{C}\mathcal{P}^+$ described by $w.f_H = f_{w(H)}$ if

$w.e_H \in \mathcal{P}^+$ and $w.f_H = -f_{w(H)}$ if $w.e_H \in \mathcal{P}^-$, where f_H denotes the basis element of $\mathbb{C}\mathcal{P}^+$ corresponding to $e_H \in \mathcal{P}^+$.

Finally, we notice that, when W is a Coxeter group, then the representation R_h for arbitrary h factorizes through the extended Coxeter group $B/(P,P)$ introduced by J. Tits in [Ti].

We give in the following table the decomposition in irreducibles of R_0, R_1 for the classical Coxeter groups of type A_n, B_n, D_n. We label as usual irreducible representations of $\mathfrak{S}_n$ by partitions of size n (with the convention that $[n]$ is the trivial representation), of W of type B_n by couples of partitions (λ, μ) of total size n, and denote $\{\lambda, \mu\}$ the restriction of (λ, μ) to the usual index-2 subgroup of W of type D_n. Recall that $\{\lambda, \mu\} = \{\mu, \lambda\}$ is irreducible if and only if $\lambda \neq \mu$.

	R_0
$A_n, n \geq 3$	$[n-1,2] + [n,1] + [n+1]$
$B_n, n \geq 4$	$([n-2,2], \emptyset) + ([n-2],[2]) + 2([n-1,1], \emptyset) + 2([n], \emptyset)$
B_3	$([1],[2]) + 2([2,1], \emptyset) + 2([3], \emptyset)$
$D_n, n \geq 4$	$\{[n-2,2], \emptyset\} + \{[n-2],[2]\} + \{[n-1,1], \emptyset\} + \{[n], \emptyset\}$
	R_1
$A_n, n \geq 3$	$[n-1,1,1] + [n,1]$
$B_n, n \geq 3$	$([n-2,1],[1]) + 2([n-1],[1])$
$D_n, n \geq 4$	$\{[n-2,1],[1]\} + \{[n-1],[1]\}$

We sketch a justification of this table. For small values of n, we prove this by using the character table. Then we use induction with respect to a natural parabolic subgroup W_0 in the same series, for which the branching rule is well-known. Restrictions of R_0 and R_1 to this parabolic subgroup are then isomorphic to the sum of the corresponding representation R_0 or R_1 of the subgroup, plus the permutation action of the reflections in W which do not belong to W_0 (this is clear for R_0, and a consequence of proposition 5.8 for R_1). The decomposition in irreducibles of this permutation representation is easy, namely $[n-1,1]+[n]$ for A_n, $([n-2],[1])+([n-2,1],\emptyset)+2([n-1],\emptyset)$ for B_n and $\{[n-2],[1]\} + \{[n-2,1], \emptyset\} + \{[n-1], \emptyset\}$ for D_n. This provides the restrictions of R_0 and R_1 to W_0. From the combinatorial branching rule it is easy to check that, for say $n \geq 5$, only the given decompositions admit these restrictions.

6. Tables for $\kappa(W)$

We compute here the value of $\kappa(W)$ for all irreducible reflection groups W. More precisely, we compute all $d \in \mathbb{Z}$ such that there exists $w \in W$ and $H \in \mathcal{A}$ with $w.e_H = \zeta e_H$ and ζ of order d. We call these integers the $\mathcal{A}$-indices of W

Recall that the group $G(de,e,r)$ for $r \geq 2$ is defined as the set of $r \times r$ monomial matrices with nonzero entries in $\mu_{de}(\mathbb{C})$, such that the product of these nonzero entries lie in $\mu_d(\mathbb{C})$.

Proposition 6.1. *The $\mathcal{A}$-indices of $W = G(de,e,r)$ are exactly the divisors of $\kappa(W)$. Moreover, $\kappa(W) = de$ if $d \neq 1$ or $r \geq 3$. If $W = G(e,e,2)$ then $\kappa(W) = 2$.*

Proof. Since $G(e,e,2)$ is a Coxeter (dihedral) group, we can assume $d \neq 1$ or $r \geq 3$. First note that the standard hermitian scalar product on $\mathbb{C}^r$ is invariant under W. We introduce the hyperplane arrangement

$$\mathcal{A}^0_{de,r} = \{z_i - \zeta z_j = 0 \mid \zeta \in \mu_{de}(\mathbb{C})$$

We have $\mathcal{A}^0_{de,r} \subset \mathcal{A}$, and the orthogonal to $H : z_i - \zeta z_j = 0$ is spanned by $e_H = e_i - \zeta^{-1} e_j$, if $e_1, \dots, e_n$ denotes the canonical basis of $\mathbb{C}^r$. Let $w \in W$. Since w is a monomial matrix, there exists $\lambda_1, \dots, \lambda_r \in \mu_{de}(\mathbb{C})$ with $\lambda_i \in \mu_{de}(\mathbb{C}), \prod \lambda_i \in \mu_d(\mathbb{C})$, and $\sigma \in \mathfrak{S}_r$ such that $w.e_i = \lambda_i e_{\sigma(i)}$. Then $w.e_H = \mu e_H$ iff $\lambda_i e_{\sigma(i)} - \lambda_j \zeta^{-1} e_{\sigma(j)} = \mu \lambda_i e_i + \mu \lambda_j e_j$. The two possibilities are $\mu = 1, \zeta = 1$ or $\mu \lambda_j = \lambda_i, \mu \lambda_i = \lambda_j \zeta^{-1}$, that is $\mu^2 = \zeta^{-1}, \mu = \lambda_i \lambda_j^{-1}$. It follows that $\mu \in \mu_{de}(\mathbb{C})$. Conversely, assume we choose $\mu \in \mu_{de}(\mathbb{C})$, and let $\zeta = \mu^{-2}$. If $r \geq 3$ we can define $w \in W$ by $\sigma = (1\ 2)$, $\lambda_2 = 1$, $\lambda_1 = \mu$, $\lambda_3 = \mu^{-1}$, $\lambda_k = 1$ for $k \geq 4$, and $w.e_H = \mu e_H$ for $H : z_1 - \zeta z_2 = 0$. We have $\mathcal{A} = \mathcal{A}^0_{de,r}$ when $d = 1$, so this settles this case and we can assume $d \neq 1$. In that case, $\mathcal{A} = \mathcal{A}^0_{de,r} \cup \mathcal{A}^+_r$, where $\mathcal{A}^+_r$ is made out the hyperplanes $H_i : z_i = 0$, whose orthogonals are spanned by the e_i. If $w.e_i = \mu e_i$ for $w \in W$ we obviously have $\mu \in \mu_{de}(\mathbb{C})$, and conversely if $\mu \in \mu_{de}(\mathbb{C})$ we can define $w \in W$ by $w.e_1 = \mu e_1, w.e_2 = \mu^{-1} e_2$ and $w.e_i = e_i$ for $i \geq 3$. It follows that in this case too the set of $\mathcal{A}$-indices is the set of divisors od de. □

By noticing that $G(2,1,r)$, $G(2,2,r)$ and $G(e,e,2)$, are Coxeter groups, this gives the following.

Corollary 6.2. *For $W = G(de,e,r)$, we have $\kappa(W) = 2$ iff W is Coxeter group, if and only if $de = 2$ or $(d,r) = (1,2)$.*

By checking out the 34 exceptional reflection groups, we prove case by case the following.

Proposition 6.3. *Let W be an irreducible complex reflection group. The set of $\mathcal{A}$-indices is exactly the set of divisors of $\kappa(W)$.*

The following table gives the value of $\kappa(W)$, where W an complex reflection group labelled by its Shephard-Todd number (ST).

ST	κ	ST	κ	ST	κ	ST	κ	ST	κ	ST	κ
4	6	10	12	16	10	22	4	28	2	34	6
5	6	11	24	17	20	23	2	29	4	35	2
6	12	12	2	18	30	24	2	30	2	36	2
7	12	13	8	19	60	25	6	31	4	37	2
8	4	14	6	20	6	26	6	32	6		
9	8	15	24	21	12	27	6	33	6		

We remark that the only non-Coxeter irreducible reflection groups with $\kappa(W) = 2$ are G_{12} and G_{24}. Like in the case of G_{12}, it is straightforward to check that it is possible to choose the 21 linear forms α_H such that the linear

map $\Phi : \mathbb{C}\mathcal{A} \to S^2V^*$ is a morphism of W-modules. This phenomenon is reminiscent of the special properties of their "root systems" in the sense of [Co]. We refer to [Sh] §2 and §4 for a detailed study of these special root systems of type G_{12} and G_{24}. In particular, convenient linear forms for G_{24} are described in [Sh], §4.1.

As a consequence of this case-by-case investigation, propositions 4.4 and 4.5 can be enhanced in the following

Theorem 6.4. *Let W be an irreducible reflection group. The linear forms α_H can be chosen such that Φ is a morphism of W-modules if and only if $\kappa(W) = 2$. This is the case exactly when W is a Coxeter group or an exceptional reflection group of type G_{12} or G_{24}.*

References

[Bo] N. Bourbaki, *Groupes et algèbres de Lie, chapitres 4,5,6*, Hermann, Paris, 1968.

[BMR] M. Broué, G. Malle, R. Rouquier, *Complex reflection groups, braid groups, Hecke algebras*, J. Reine Angew. Math. **500**, 127-190 (1998).

[Co] A.M. Cohen, *Finite complex reflection groups*, Ann. E.N.S. **9**, 379-436 (1976).

[Ma04] I. Marin, *Irréductibilité générique des produits tensoriels de monodromies*, Bull. Soc. Math. Fr. **132**, 201-232 (2004).

[Ma07] I. Marin, *Krammer representations for complex braid groups*, preprint arXiv:0711.3096 (v2).

[OT] P. Orlik, H. Terao, *Arrangements of hyperplanes*, Springer, Berlin, 1992.

[ST] G.C. Shephard, J.A. Todd, *Finite unitary reflection groups*, Canad. J. Math **6**, 274-304 (1954).

[Sh] J.-Y. Shi, *Simple root systems and presentations for certain complex reflection groups*, Comm. Alg. **33**, 1765-1783 (2005).

[Ti] J. Tits, *Normalisateurs de tores I : Groupes de Coxeter étendus*, J. Algebra **4**, 96-116 (1966).